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THE SOCIAL COSTS OF MONOPOLY AND REGULATION A GAME THEORETIC ANALYSIS

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I. INTRODUCTION

Posner (1975) made a significant contribution to the theory of regulation by pointing out that the existence of an opportunity to obtain monopoly profits will attract resources into efforts to obtain monopolies, and the opportunity costs of those resources are social costs of monopoly too. This general insight is undoubtedly true and is very useful in analyzing seemingly diverse types of behavior. In the process of such analysis several specific technical questions arise very naturally; Posner speculates on their answers but does not prove his assertions. The purpose of this paper is to explicitly address these unanswered questions. In general we wish to predict the behavior we would observe when a number of different firms compete for the right to earn a monopoly profit by receiving some sort of franchise. Would they spend more or less than the total potential monopoly profit? Posner speculates

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that they would collectively spend precisely the potential profit.

> If ten firms are vying for a monopoly having a present value of \$1 million, and each of them has an equal chance of obtaining it and is risk neutral, each will spend \$100,000 (assuming constant costs) on trying to obtain the monopoly. Only one will succeed and his costs will be much smaller than the monopoly profits, but the total costs of obtaining the monopoly--counting losers' expenditures as well as winners--will be the same as under certainty.[1]

How would this situation change if the current monopolist were to have an advantage over potential entrants at regulatory hearings to (possibly) reassign the franchise? Would firms collectively spend more or less? Would the current monopolist outspend or underspend the potential entrants? Posner speculates that the tendency for entrants to spend less because of their lower chances would be counteracted by their desire to spend more because owning the monopoly is now worth more, and that, therefore, a situation where the current monopolist has an advantage should not be any different.

Technical analysis shows Posner's assertion that total expenditures of competing firms equals the franchise profit to be less general than he claimed. It is critical that we additionally assume that competitive pressures of some sort (such as entry) drive the value of the game for all firms to zero. In this case the long-run expected expenditures of firms will equal the franchise profit. However, expenditures may exceed, equal, or fall short of the

[1] Posner (1975) p. 812

franchise profit in any given period. If in addition, firms are identical and in a symmetric equilibrium, combined expenditures of firms will be precisely the franchise profit every period. Without the assumption that all values are zero, we can only say that the long-run expected expenditures of firms will not exceed the franchise profit. As before, expenditures may exceed, equal, or fall short of the franchise profit in any given period. For the case of identical firms in a symmetric equilibrium, we can prove that combined expenditures on the part of entrants will never exceed the franchise profit in any period.

In a more specific model where entrants are identical and risk neutral and a Nash equilibrium concept is used, it is possible to prove that differing degrees of advantage conferred upon the current monopolist result in differing behavior on the part of the participating firms and in differing aggregate expenditures. An increased advantage to the current monopolist tends to decrease aggregate expenditures if firms discount future profits highly, if there are many potential entrants, or if the current monopolist's advantage is already quite high. On the other hand, an increased advantage to the current monopolist tends to increase aggregate expenditures if firms discount future profits very little, if there are few potential entrants, or if the current monopolist's advantage is already quite low.

Furthermore, in this model, an increase in the competitive pressures in the form of an increased number of potential entrants does not drive the aggregate value of the game to zero even in the limit. Therefore, combined expenditures remain a discrete distance below the franchise profit even in the limit. This tendency becomes more pronounced as the advantage to the monopolist is increased. At least in some cases, therefore, we would never expect combined expenditures by firms to equal the franchise profit.

II. THE GENERAL CASE

We wish to model the idealized situation where n firms compete for the right to operate a monopoly franchise which generates profits. At the start of each period, a government agency assigns rights to the franchise for that period. Firms spend money attempting to influence this decision.

Formally, we construct an n-person infinite period game. There are n states; let state j be the state where firm j is currently the monopolist. Let x_{ij} be the amount of money spent by firm i when state j occurs. The state space and firms' strageties are stationary. They are not dependent on t. A firm's probability of obtaining the franchise clearly will depend upon its and others' lobbying expenditures. It may also depend upon what state the world is in. For example, a firm's chances of success may be greater if it is the monopolist because it now has greater knowledge and expertise or because it has established a relationship with its regulators. Let f_{ij} be the probability of firm i succeeding in state j; f_{ij} is a function of $(x_{1j},\ldots,x_{nj})$. It must be true for every $(x_{1j},\ldots,x_{nj})$ that

$$\sum_{i=1}^{n} f_{ij} = 1 \quad \forall\ j \tag{1}$$

$$f_{ij} \geq 0 \quad \forall\ i,j. \tag{2}$$

Let π be the profits that the successful firm earns in a period by operating the franchise.

Firm i selects the strategy vector $(x^*_{i1},\ldots,x^*_{in})$. That is, firm i spends x^*_{ij} if state j occurs. Let θ_{ij} be firm i's probability of success in state j with the given strategies:

$$\theta_{ij} = f_{ij}(x^*_{1j},\ x^*_{2j},\ldots,x^*_{nj}) \tag{3}$$

Let θ be the matrix with (i,j) entry of θ_{ij}.

$$\theta = [\theta_{ij}] \tag{4}$$

By (1) and (2) θ is a stochastic matrix. The associated stochastic process is the one that determines which firm will be the monopolist for each period.

Such a matrix always has a steady state solution--a vector $\gamma = [\gamma_1,\ldots,\gamma_n]$ such that

$$\text{(i)} \quad \gamma_i \leq 0 \quad \forall \quad i=1, \tag{5}$$

$$\text{(ii)} \quad \sum_{i=1}^{n} \gamma_i = 1 \tag{6}$$

$$\text{(iii)} \quad \gamma = \theta\gamma \tag{7}$$

The steady state may not be unique; as well, the stochastic process may actually converge to some sort of cycle. However, at a minimum, the average of the cycle which the stochastic process converges to equals one of the steady states. Therefore, one of the steady states describes the long-run average probability distribution which the stochastic process will exhibit. (Which steady state does this depends on the initial point.) Long-run expected values for this process are therefore calculated by using one of the steady state distributions. See Gantmacher (1960) for a complete discussion of these points.

Before stating and proving the major theorem of this section, more notation needs to be introduced. Let θ^t_{ij} be the probability of firm i becoming the monopolist in period t given that firm j is currently the monopolist. That is, θ^t_{ij} is the (i,j) entry of θ^t. Let R^t_{ij} be the expected profit to firm i in period t given that the world is currently in state j. These returns can be defined recursively as follows:

$$R^1_{ij} = \theta_{ij}\pi - x^*_{ij} \tag{8}$$

$$R^t_{ij} = \sum_{k=1}^{n} \theta^{t-1}_{kj} R^1_{ik} \qquad t = 2,3,... \tag{9}$$

I will assume that all firms calculate the value of the game by summing discounted expected profits, firm i using the discount rate c_i. Let V_{ij} be the value of the game to firm i in the state j.

$$V_{ij} = \sum_{t=1}^{\infty} c_i^{t-1} R^t_{ij} \tag{10}$$

Finally, let S_j be the surplus of π over total firm expenditures which occurs in state j.

$$S_j = \pi - \sum_{i=1}^{n} x^*_{ij} \tag{11}$$

The major theorem of this section is that the long-run expected surplus that this game generates is a nonnegative weighted sum of the values $\{V_{ij}\}$.

<u>Theorem 1</u>:

Let γ be a steady state solution to θ. Then

$$\sum_{j=1}^{n} \gamma_j S_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_j (1 - c_i) V_{ij} \tag{12}$$

<u>Proof</u>:

See Appendix □

The value of the game to every player must always be nonnegative if we assume that the strategy of doing nothing at zero cost is available to each player. Therefore, the RHS of (12) is nonnegative; therefore, so is the LHS. Therefore, the long-run expected surplus of π over total expenditures is nonnegative. However, the surplus in any particular state could be negative. That is, in the general case we cannot predict anything about the surplus in any particular period; we can only predict that in the long run the average surplus will be nonnegative. Suppose, however, that the surplus generated was the same in every state. (This would occur, for example, in a symmetric equilibrium of identical firms.) Then the surplus every period would be nonnegative.

Finally, consider the case where the value of the game is zero for every player in every state. This might occur, for example, if there were many firms of each type and entry occurred. Then the firms collectively spend on average π. If there is a symmetric equilibrium of identical firms they would then spend precisely π every period in accord with Posner's prediction.

III. A SPECIAL CASE

We can make a surprising number of predictions about the general case where all firms have different return functions, use different discount rates and no equilibrium concept is specified. However, the problem of identifying the effects of increasing the current monopolist's advantage is not tractable in such a setting. This problem does become tractable in the specific setting which is presented in this section. As well the model of this section provides a useful concrete example of the propositions proved for the more abstract model.

For expositional convenience assume that there are n + 1 firms (instead of r), 1 current monopolist and n potential entrants. Let f_{ij} be given by

$$f_{ij}(0,\ldots,0) = 0 \tag{13-a}$$

and by

$$f_{ij}(x_{1j},\ldots,x_{n+1,j}) = \begin{cases} \dfrac{\beta x_{ij}}{\beta x_{jj} + \sum_{k\neq j} x_{kj}}, & i = j \\[2ex] \dfrac{x_{ij}}{\beta x_{jj} + \sum_{k\neq j} x_{kj}}, & i \neq j \end{cases} \tag{13-b}$$

for every $(x_{1j},\ldots,x_{n+1,j}) \neq 0$,

where β is some real number greater than or equal to 1. Each firm's chance of obtaining the franchise is simply the proportion of total lobbying expenditures that it accounts for weighted by an advantage for the current monopolist. All potential entrants are treated the same. Each firm faces the same return function if it becomes the monopolist. This is, therefore, in some sense the simplest specification of a case where the monopolist has an advantage. As well, assume that each firm uses the same discount rate, c.

We will use the Nash equilibrium concept.

<u>Definition</u>:

The strategy vectors $\{x_1^*,\ldots,x_{n+1}^*\}$ are equilibrium strategies if for every i and j, x_i^* satisfies

$$V_{ij}(x_1^*,\ldots,x_i^*,\ldots,x_{n+1}^*) = \sup_{x_i \in R_+^{n+1}} V_{ij}(x_1^*,\ldots,x_i,\ldots,x_{n+1}^*) \tag{14}$$

That is, x_i^* must maximize $(V_{i1},\ldots,V_{i,n+1})$ given others' behavior. Note that the domain of x_i is R_+^{n+1} (where $R_+ = [o,\infty)$). In particular, the firm always has the option of doing nothing at zero cost and zero return. Recall from (13) that our return functions are particularly simple--a firm's probability of winning is not affected by which of the others is the monopolist so long as they all spend the same. Because of this we might hope for a particularly simple sort of equilibrium to occur; each firm's strategy is one number, x_e, if it is an entrant and another number, x_m, if it is the monopolist. We will call this a symmetric equilibrium.

<u>Definition</u>:

An equilibrium $\{x_i^*\}_{i=1}^{n+1}$ is a symmetric equilibrium if there exist two numbers x_e and x_m such that for every i and j

$$x_{ij}^* = \begin{cases} x_e, & i \neq j \\ x_m, & i = j \end{cases} . \tag{15}$$

The major result of this section is the constructive proof of the existence and uniqueness of such a symmetric equilibrium. Since each player has the same strategy and it only varies as he is the entrant or monopolist, it will be seen that all of the variables indexed by (i,j) will only assume two values. We will employ the notational convenience of indexing them by "e" and "m" for the state of being the entrant or being the monopolist.

<u>Theorem 2</u>:

<u>Theorem 2</u>:

The unique symmetric equilibrium is

$$x_e = \frac{\beta n \pi}{(\beta n + 1)^2 - cn^2(\beta - 1)^2 - 2cn(\beta - 1)} \tag{16}$$

$$x_m = \left(n - \frac{n-1}{\beta}\right) x_e . \tag{17}$$

Other variables assume the following values (uniquely):

$$\theta_e = \frac{1}{\beta n + 1} \tag{18}$$

$$\theta_m = \frac{\beta n + 1 - n}{\beta n + 1} \tag{19}$$

$$R_e^1 = \frac{\pi}{\beta n + 1} - x_e \tag{20}$$

$$R_m^1 = \frac{\pi(\beta n + 1 - n)}{\beta n + 1} - x_m \tag{21}$$

$$V_e = \frac{c\,\theta e}{D} R_m^1 + \frac{(1 - c) + cn\,\theta e}{D} R_e^1 \tag{22}$$

$$V_m = \frac{(1 - c) + c\,\theta e}{D} R_m^1 + \frac{cn\,\theta e}{D} R_e^1 \tag{23}$$

where

$$D = (1 - c)\{(1 - c) + c\,\theta e(1 + n)\} \tag{24}$$

<u>Proof</u>:

See Appendix □

Before discussing the substantive qualties of the model one comment on uniqueness is in order. For the case where

$c = 0$, that is where the world ends after the first period, the symmetric equilibrium is in fact the unique equilibrium. Therefore, although I was unable to prove that the symmetric equilibrium is the unique equilibrium for the general case, my solution is a generalization of the unique solution for the one period case.

It is interesting to note that although V_e and V_m are always positive, R_e^1 may be negative. (R_m^1 must always be positive.) That is, entrants may play the game expecting to make a short-run loss and recoup it in the future by becoming the monopolist. Note also that V_e and V_m are convex combinations of $R_m^1/(1 - c)$ and $R_e^1/(1 - c)$. That is, we can write

$$V_e = \frac{\alpha\, R_m^1 + (1 - \alpha)R_e^1}{(1 - c)} \tag{25}$$

$$V_m = \frac{\alpha\, R_e^1 + (1 - \alpha)R_m^1}{(1 - c)} \tag{26}$$

where

$$\alpha = \frac{c\ \theta e}{(1 - c) + c(1 + n)\theta e}. \tag{27}$$

Therefore, being an entrant is equivalent to receiving the gamble (R_m^1, R_e^1) with probability $(\alpha, 1 - \alpha)$ year after year. Similarly, being the monopolist is equivalent to receiving the gamble (R_m^1, R_e^1) with probabilities $(1 - \alpha, \alpha)$ every year.

The theorems of the abstract model can be interpreted in our setting. Equation (12) becomes (28).

Corollary 1:

$$\pi - n\, x_e - x_m = (1 - c)(n\, V_e + V_m) \tag{28}$$

Proof:

This is an immediate corollary of theorem 1. It can also be derived from Theorem 2 by performing the required algebra. □

Since all firms are treated the same, in the long run each is equally likely to be the monopolist; that is $\gamma_j = 1/(n + 1)$ for every j. As well, due to the symmetric nature of the equilibrium, the surplus, S_j, is the same in every state. This, therefore, is the special case discussed in section II where the surplus is definitely nonnegative every period.

It was also suggested in section II that entry of more players into the game might drive the aggregate value of the game, and thus by (12) also the expected surplus, to zero. We can investigate this question in the specific model of this section. Incidentally, this question also suggests that we might formally define an entry equilibrium into the game. Since V_e is always positive (see the proof of Theorem 2) we must assume the existence of some fixed cost of entry into the game varying across firms to generate an entry equilibrium. Interpret this fixed cost as an organization cost. In this context we are asking if the surplus of π over firms' expenditures goes to zero when fixed organization costs are low enough for enough firms.

Corollary 2:

As n goes to infinity, the symmetric equilibrium converges to the following:

$$\lim_{n\to\infty} x_e = 0 \tag{29}$$

$$\lim_{n\to\infty} n\, x_e = \frac{\beta}{\beta^2 - c(\beta - 1)^2}\,\pi \qquad (30)$$

$$\lim_{n\to\infty} x_m = \frac{\beta - 1}{\beta^2 - c(\beta - 1)^2}\,\pi \qquad (31)$$

Proof:

Simply perform the required operations. □

The case where the monopolist has no advantage ($\beta = 1$) might be used as a base case. In this case, as n grows expenditures by any particular firm go to zero, but expenditures by the aggregate converge upwards to π. That is, entry causes aggregate expenditures to converge to π and the aggregate value of the game to converge to zero as hypothesized. However, the situation changes as the advantage to the monopolist grows. Expenditures by the current monopolist do not disappear in the limit. Furthermore, aggregate expenditures are less than π. That is, even in the limit π exceeds aggregate expenditures and the aggregate value of the game is positive. (It is possible to prove that V_e goes to 0 even in this case, however). Therefore, entry might not always cause lobbying expenditures to equal π even if fixed organization costs are low. The tendency for a positive surplus to persist in the limit is more pronounced when the current monopolist is given a larger advantage.

The question of the general effect of the size of β on the surplus remains to be investigated.

Corollary 3:

(i) $$\frac{D\,x_e}{D\beta} = \frac{-n^3[(1 - c)\beta^2 + c] + 2cn^2 + n}{*^2}\,\pi \qquad (32)$$

(ii) $$\frac{Dx_m}{D\beta} = \frac{(\beta n + 1)(1 - c)(2n - (\beta n + 1)) + c(n^2 + 1)}{*^2}\,n^2\pi \qquad (33)$$

(iii) $$\frac{D}{D\beta}(nx_e + x_m) = \frac{(1 + \beta n)2n(1 - \beta) + c(2 + 2\beta n + 2n^2\beta^2 - 2\beta n^2)}{*^2}\,n^2\pi \qquad (34)$$

where

$$* = (\beta n + 1)^2 - cn^2(\beta - 1)^2 - 2cn(\beta - 1) \qquad (35)$$

Proof:

Simply perform the required operations. □

The firm receives two conceptually different rewards from winning the franchise. In the short run, it receives the franchise profit that period. From a longer-run perspective, its expected return for the rest of the game is also increased due to the monopolist's advantage. Therefore increasing β increases the value of winning by by increasing the long-run expected returns. Simply considering this factor might lead us to expect that increases in β should result in increases in expenditures of all firms. That is, an increased reward should induce increased efforts to obtain it. However, two factors complicate this. First, changing β also changes the marginal return of the monopolist's and entrants' expenditures. In particular, the entrants' chances of winning are decreased for any given expenditure.

This might induce them to spend less as β rises. Second, the firms may respond to any change in the others' expenditures. However, certainly as the future matters more (i.e., c is larger), we might expect our initial hypothesis to be more applicable since the effect of increasing β on the value of winning becomes larger.

The signs of $Dx_e/D\beta$ and $Dx_m/D\beta$ are somewhat reflective of this intuition. When c is a neighborhood of 0, $Dx_e/D\beta$ is always negative. However, as c increases beyond this point, the sign becomes ambiguous. The reverse is true for $Dx_m/D\beta$. When c is in a neighborhood of 1, its sign is positive; as c decreases beyond this point, the sign becomes ambiguous. Finallly, as n increases, an entrant's probability of winning goes to zero. Therefore, we would expect the long-run factor to become unimportant and for the entrant to behave in a fashion similar to his behavior when c equals zero. This in fact occurs. It is easy to see that for every (β,c) there exists an n^o such that

$$n \geq n^o \Rightarrow \frac{Dx_e}{D\beta} < 0. \tag{36}$$

The sign of the derivative of total expenditure per period with respect to β, $\frac{D}{D\beta}(nx_e + x_m)$, behaves even more intuitively. First, as for previous cases, the derivative is more likely to be positive as c increases, thereby increasing the effect of the long-run factor.

Corollary 4:

Let $$\delta(\beta,n) = \frac{\beta n^2(\beta - 1) + \beta n - n}{\beta n^2(\beta - 1) + \beta n + 1}. \tag{37}$$

(i) $$c \begin{matrix} < \delta \\ = \delta \\ > \delta \end{matrix} \Leftrightarrow \frac{D(nx_e + x_m)}{D\beta} \begin{matrix} < \\ = \\ > \end{matrix} 0. \tag{38}$$

(ii) For every $(\beta,n) \in (1,\infty) \times \{1,2,3,...\}$

$$0 \leq \delta(\beta,n) < 1. \tag{39}$$

(iii) $$\frac{D\delta}{D\beta} > 0, \quad \frac{D\delta}{Dn} > 0. \tag{40}$$

Proof:

Obvious. □

That is, for every (β,n) we can divide the interval [0,1] into two sections. If c is in the left section, expenditures fall (the surplus rises) as β rises; if c is in the right hand section, expenditures rise (the surplus falls) as β rises. Increasing β or increasing n moves this cutoff point to the right. Increasing β is initially more likely to cause a rise in expenditures; however, as it is continually increased it will eventually result in falls of total expenditures. The intuition behind the sign of $D\delta/Dn$ is the same as used previously; as n increases, the probability of a given firm winning decreases and therefore the long-run effect matters less.

Two other results in accord with the intuition of (iii) of Corollary 4 are:

Corollary 5:

(i) For every (n,c) there exists a β^o such that

$$\beta \geq \beta^o \Rightarrow \frac{D}{D\beta}(nx_e + x_m) < 0. \tag{41}$$

(ii) For every (β,c) there exists an n^o such that

$$n \geq n^o \Rightarrow \frac{D}{D\beta}(nx_e + x_m) < 0. \tag{42}$$

Proof:

Obvious. □

IV. CONCLUSION

In general, the long-run expected surplus of the franchise profit over combined expenditures by the firms equals the long-run expected aggregate value of the game. Since the latter will generally be nonnegative, so is the former. If competitive pressures drive the latter to zero, the former is also zero. If in addition all firms are identical and the equilibrium is symmetric, we have Posner's assertion -- combined expenditures by firms will precisely equal the franchise profit every period. However, there is some question as to whether we can reasonably expect competitive pressures to drive the aggregate value of the game to zero. In the simple example of section III, increases in the number of potential entrants does not eliminate a positive aggregate value of the game, even in

the limit. This tendency becomes more pronounced as the advantage to the current monopolist is increased. In this model, differences in the advantage of the monopolist result in differing behavior. An increased advantage to the current monopolist tends to decrease (increase) aggregate expenditures if firms discount future profits highly (very little), if there are many (few) potential entrants, or if the current monopolist's advantage is already quite high (low).

Possibilities exist for applying and testing this model in real situations. The regulatory process is generally characterized by periodic review and (possible) reassignment of the franchise. Firms are often required to report their expenditures associated with participation in this process. The franchise profit is also public information. Other phenomena may also be investigated with variants of this model. Many political processes fit very naturally into this framework. For example, political parties battling for electoral success over an infinite horizon often desire to win a particular election not only because of the immediate reward but also because their chances of winning subsequent elections are increased. The same type of reasoning applies to presidential candidates competing in a succession of primaries to win their party's nomination. I am currently working on these applications.

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APPENDIX

Proof of Theorem 1:

The value functions must all satisfy

$$V_{ij} = R^1_{ij} + c_i \sum_{k=1}^{n} \theta_{kj} V_{ik}. \tag{A-1}$$

Therefore

$$\sum_{j=1}^{n} \gamma_j V_{ij} = \sum_{j=1}^{n} \gamma_j R^1_{ij} + c_i \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \gamma_j \theta_{kj} \right) V_{ik} \tag{A-2}$$

$$= \sum_{j=1}^{n} \gamma_j R^1_{ij} + c_i \sum_{k=1}^{n} \gamma_k V_{ik}. \tag{A-3}$$

The last step is by (7). Now reorganize.

$$(1 - c_i) \sum_{j=1}^{n} \gamma_j V_{ij} = \sum_{j=1}^{n} \gamma_j R^1_{ij}. \tag{A-4}$$

Summing over i yields

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (1 - c_i) \gamma_j V_{ij} = \sum_{j=1}^{n} \gamma_j \left(\sum_{i=1}^{n} R^1_{ij} \right). \tag{A-5}$$

Substitute (8) into (A-5).

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (1 - c_i) \gamma_j V_{ij} = \sum_{j=1}^{n} \gamma_j \left(\sum_{i=1}^{n} \theta_{ij} \pi - x^*_{ij} \right). \tag{A-6}$$

Then by (1) we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (1 - c_i)\gamma_j V_{ij} = \sum_{j=1}^{n} \gamma_j(\pi - \sum_{i=1}^{n} x^*_{ij}) \tag{A-7}$$

$$= \sum_{j=1}^{n} \gamma_j S_j. \tag{A-8}$$

□

Proof of Theorem 2:

For $\{x^*_j\}_{j=1}^{n+1}$ to be an equilibrium, it is necessary and sufficient that x^*_j is an optimal strategy for firm i when it takes the others' strategies as given. A necessary and sufficient condition for this latter event to hold is that the relevant functional equation of dynamic programming be satisfied [Denardo, 1967].

Choose an arbitrary firm i. Fix the other firms strategies at $\{x^*_j\}_{j\neq i}$. Then define (n + 1) real valued functions of a real variable.

First for $i \neq j$, let α_{ij} be

$$\alpha_{ij}(x) = \begin{cases} \beta x^*_{jj} + x + \sum\limits_{\substack{k\neq i\\ k\neq j}} x^*_{kj}, & \beta x^*_{jj} + x + \sum\limits_{\substack{k\neq i\\ k\neq j}} x^*_{kj} \neq 0 \\ 1, & \beta x^*_{jj} + x + \sum\limits_{\substack{k\neq i\\ k\neq j}} x^*_{kj} = 0 \end{cases} \tag{A-9a}$$

Then for i = j, let α_{ij} be

$$\alpha_{ii}(x) = \begin{cases} \beta x + \sum\limits_{k\neq i} x^*_k, & \beta x + \sum\limits_{k\neq i} x^*_{ki} \neq 0 \\ 1, & \beta x + \sum\limits_{k\neq i} x^*_{ki} = 0 \end{cases} \tag{A-9b}$$

Now define (n + 1) real valued functions over R^{n+2}.

$$g_{ij}(x, y_1, \ldots, y_{n+1}) = \begin{cases} \dfrac{x}{\alpha_{ij}(x)} \pi - x + \dfrac{cx}{\alpha_{ij}(x)} y_i + \dfrac{c\beta x^*_{jj}}{\alpha_{ij}(x)} y_j + \sum\limits_{\substack{k\neq i\\ k\neq j}} \dfrac{cx^*_{kj}}{\alpha_{ij}(x)} y_k, & j \neq i \\ \dfrac{\beta x}{\alpha_{ij}(x)} \pi - x + \dfrac{c\beta x}{\alpha_{ij}(x)} y_i + \sum\limits_{k\neq i} \dfrac{cx^*_{kj}}{\alpha_{ij}(x)} y_k, & j = i \end{cases} \tag{A-10}$$

Then a nonnegative x^*_i is optimal for firm i given the others' behavior if and only if there exist n + 1 real numbers ($V_{i1}, \ldots, V_{i,n+1}$) such that (A-11) and (A-12) are satisfied. Furthermore, V_{ij} is the value of the game to firm i at state j.

$$V_{ij} = \sup_{x\varepsilon[0,\infty)} g_{ij}(x, V_{i1}, \ldots, V_{i,n+1}) \tag{A-11}$$

for every j = 1,..., n + 1.

$$g_{ij}(x^*_{ij},V_{i1},\ldots,V_{i,n+1}) = \sup_{x\varepsilon[0,\infty)} g_{ij}(x,V_{i1},\ldots,V_{i,n+1})$$

for every $j = 1,\ldots,n + 1$.

To prove existence, it is therefore sufficient to substitute the following

$$x^*_{kj} = \begin{cases} x_e, & k \neq j \\ x_m, & k = j \end{cases} \tag{A-13}$$

$$V_{ij} = \begin{cases} V_e, & i \neq j \\ V_m, & i = j \end{cases} \tag{A-14}$$

into (A-11) and (A-12) and verify that (A-11) and (A-12) are true. Since the selection of i was arbitrary, verification of optimality for one i is sufficient to guarantee optimality for all i's. This is straightforward. (At least conceptually if not algebraically!) However, we do not obtain uniqueness in this fasion. As well, such a method gives no idea how I choose the values for x_{ij} and V_{ij}. Therefore, instead, I shall outline a more constructive proof.

A nonnegative strategy (x_e,x_m) with associated values (V_e,V_m) is a symmetric equilibrium if and only if (A-11) and (A-12) are satisfied. We can rewrite (A-11) and (A-12) in simplier form because the strategies are so simple. Let

$$\alpha_e(x) = \begin{cases} \beta x_m + x + (n-1)x_e, & \beta x_m + x + (n-1)x_e \neq 0 \\ 1, & \beta x_m + x + (n-1)x_e = 0 \end{cases} \tag{A-15a}$$

$$\alpha_m(x) = \begin{cases} \beta x + nx_e, & \beta x + nx_e \neq 0 \\ 1, & \beta x + nx_e = 0 \end{cases} \tag{A-15b}$$

Then rewrite (A-10) as

$$g_e(x,y_e,y_m) = \frac{x}{\alpha_e(x)}\,\pi - x + \frac{cx}{\alpha_e(x)}\,y_m + \frac{c(\beta x_m + (n-1)x_e)}{\alpha_e(x)}\,y_e \tag{A-16a}$$

$$g_m(x,y_e,y_m) - \frac{\beta x}{\alpha_m(x)}\,\pi - x + \frac{c\beta x}{\alpha_m(x)}\,y_m + \frac{cn\,x_e}{\alpha_m(x)}\,y_e. \tag{A-16b}$$

Then (A-11) and (A-12) become

$$V_e = g_e(x_e,V_e,V_m) \tag{A-17}$$

$$V_m = g_m(x_m,V_e,V_m) \tag{A-18}$$

$$g_e(x_e,V_e,V_m) = \sup_{x\varepsilon[0,\infty)} g_e(x,V_e,V_m) \tag{A-19}$$

$$g_m = (x_m,V_e,V_m) = \sup_{x\varepsilon[0,\infty)} g_m(x,V_e,V_m). \tag{A-20}$$

That is, a nonnegative pair (x_e,x_m) is a symmetric equilibrium if and only if there exist numbers V_e and V_m such that (A-17) through

(A-20) are satisfied.*

We thus have four equations in four unknowns -- x_e, x_m, V_e and V_m. Proving the theorem now amounts to proving the existence and uniqueness of a solution to these equations.

First, I will show that the solution to x_e and x_m is necessarily interior; both x_e and x_m are positive. This will allow me to use the necessary conditions for an interior extremum. Suppose that both x_e and x_m equal 0. Then (A-17) and (A-18) become

$$V_e = 0 \tag{A-21}$$

$$V_m = 0 \tag{A-22}$$

Then (A-19) and (A-20) are both the same equation.

$$0 = \sup_{x\varepsilon[0,\infty)} \begin{cases} \pi - x, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{A-23}$$

Equation (A-23) is, of course, not true. Therefore, both x_e and x_m cannot be zero. Suppose that x_e equals 0. Then (A-17) and (A-18) require that V_e be 0 and V_m be $(\pi - x_m)/(1 - c)$. Then (A-20) becomes

*Technically, for the "only if" to be true, we must prove that V_{ij} will only assume two values for firm i under (A-13), one for when i is the entrant and one for where i is the monopolist. This is easily seen to be true by using the direct definition of the value function (10).

$$\frac{\pi - x_m}{1 - c} = \sup_{x\varepsilon[0,\infty)} \begin{cases} \pi - x + \frac{c}{1 - c}(\pi - x_m), & x \neq 0 \\ \frac{c}{1 - c}(\pi - x_m), & x = 0 \end{cases} \tag{A-24}$$

This clearly is only true if $x_m = 0$. (Otherwise choosing $x < x_m$ yields a supremum larger than $\pi - x_m/(1 - c)$.) Therefore x_m must be zero as well which cannot be by previous considerations. Similarly, we derive a contradiction if we assume x_m equals 0.

Therefore, the first order conditions for an interior maximum are necessary for (A-19) and (A-20) to be true. If we differentiate g_e and g_m with respect to x twice we find a sufficient condition for both to be strictly concave in x is $V_m \geq V_e$. It is easy to prove that this is in fact the case. From (A-19) we have that

$$V_e \geq g(0,V_e,V_m) \tag{A-25}$$

$$= cV_e \tag{A-26}$$

Since $0 \leq c < 1$, V_e must be nonnegative. Then by (A-20),

$$V_m \geq g(0,V_e,V_m) \tag{A-27}$$

$$= cV_e \tag{A-28}$$

Since V_e is nonnegative, $V_m \geq V_e$.

Therefore, we can replace (A-19) and (A-20) by the necessary conditions for an interior maximum. They become, respectively, (A-29) and (A-30). Note that $\alpha_e(x_e)$ equals $\alpha_m(x_m)$. We will simply write α for this function of x_m and x_e.

$$V_m - V_e = \frac{\alpha^2 - (\beta x_m + (n-1)x_e)\pi}{c(\beta x_m + (n-1)x_e)} \qquad \text{(A-29)}$$

$$V_m - V_e = \frac{\alpha^2 - \beta n\, x_e\, \pi}{c\ \beta n\ x_e} \qquad \text{(A-30)}$$

By equating the RHS of (A-29) and (A-30), it is now easy to prove that any pair (x_e, x_m) satisfies (A-29) and (A-30) only if

$$x_m = \left(n - \frac{n-1}{\beta}\right)x_e \qquad \text{(A-31)}$$

That is, the RHS of (A-29) equals the RHS of (A-30) if and only if (A-31) is true.

Taking stock for a moment, we now have that (x_e, x_m, V_e, V_m) is a solution to our original equations if and only if (A-17), (A-18), (A-30) and (A-31) are satisfied. By algebraic manipulation, (A-17) and (A-18) can be seen to imply that

$$V_m - V_e = \frac{(\beta x_m - x_e)\pi + \alpha(x_e - x_m)}{\alpha - c(\beta x_m - x_e)} \qquad \text{(A-32)}$$

Therefore, any x_e and x_m which are part of the solution must satisfy

$$\frac{(\beta x_m - x_e)\pi + \alpha(x_e - x_m)}{\alpha - c(\beta x_m - x_e)} = \frac{\alpha^2 - \beta n\, x_e\, \pi}{c\ \beta n\ x_e} \qquad \text{(A-33)}$$

In fact, (x_e, x_m) is a symmetric equilibrium if and only if (A-33) and (A-31) are satisfied. This is clearly necessary. To see sufficiency, suppose that (x_e, x_m) does satisfy (A-33) and (A-31). Then since (A-17) and (A-18) are two linear equations in V_e and V_m, we can always find a solution for V_e and V_m which satisfies (A-17) and (A-18). Then since the solution satisfies (A-32) and since (x_e, x_m) satisfy (A-33), we know that (A-30) is satisfied as well.

There is precisely one solution to (A-33) and (A-31). It is

$$x_e = \frac{\beta n \pi}{(\beta_{n+1})^2 - cn^2(\beta - 1)^2 - 2cn(\beta - 1)} \qquad \text{(A-34)}$$

$$x_m = \left(n - \frac{n-1}{\beta}\right)x_e \qquad \text{(A-35)}$$

It is easy to verify that both numbers are positive. Now we simply substitute these back into (A-17) and (A-18) to determine V_e and V_m. Equations (A-17) and (A-18) can be viewed as two linear equations in two unknowns -- V_e and V_m. The determinant turns out to be nonzero and so there is precisely one solution, that given in the statement of Theorem 2. □